Trajectory Tracking and Balancing of Triple Pendulum on a Cart

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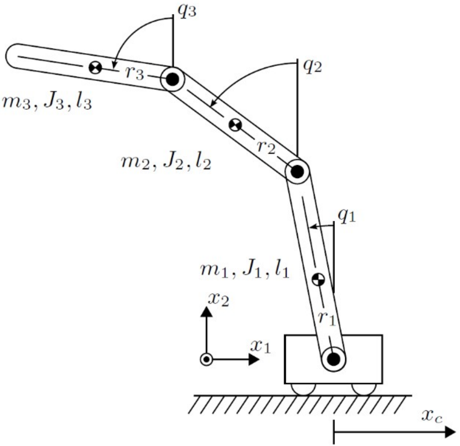
Triple Pendulum on a Cart

- Passive triple pendulum on an actuated cart.
- Configuration vector: $q = [x_c,\ \vartheta_1,\ \vartheta_2,\ \vartheta_3]^T$.
- x_c: cart position.
- ϑ_i: absolute orientation of joint i w.r.t the vertical axis.
- **Target configuration:** Upward vertical equilibrium (open-loop unstable).



Euler-Lagrange Formulation

Dynamics derived from the Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{U}$ and Rayleigh dissipation function $\mathcal{R}$:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial \mathcal{R}}{\partial \dot{q}_i} = \tau_i, \quad i = 0, \ldots, 3$$

where $i = 0$ is the cart and $i = 1, 2, 3$ are the revolute joints.

Energies:

$$\mathcal{T} = \frac{1}{2} m_0 \dot{x}_c^2 + \frac{1}{2} \sum_{i=1}^{3} \left(m_i \dot{p}_{ci}^T \dot{p}_{ci} + J_i \omega_i^2 \right)$$

$$\mathcal{U} = G_0^T (m_1 p_{c1} + m_2 p_{c2} + m_3 p_{c3})$$

$$\mathcal{R} = \frac{1}{2} d_1 \dot{\vartheta}_1^2 + \frac{1}{2} d_2 (\dot{\vartheta}_2 - \dot{\vartheta}_1)^2 + \frac{1}{2} d_3 (\dot{\vartheta}_3 - \dot{\vartheta}_2)^2$$

Dynamic Model

Resulting in standard nonlinear dynamic form:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = B\tau$$

- $M(q) \in \mathbb{R}^{4\times 4}$: symmetric and positive definite inertia matrix.
- $C(q,\dot{q}) \in \mathbb{R}^{4\times 4}$: Coriolis matrix.
- $D \in \mathbb{R}^{4\times 4}$: damping matrix.
- $g(q) \in \mathbb{R}^{4}$: gravity vector.
- $B = [1,0,0,0]^T$: input selection vector (only cart is actuated).

Overview of Control Approaches

Two distinct control problems are presented:

1. **Periodic Orbit Stabilization (Jahn et al.)**
   - **Task:** execute periodic orbits for the links.
   - **Methodology:** BVP to find nominal periodic orbit and TV-LQR for local tracking.
2. **Cascaded-EIC (Han & Yi)**
   - **Task:** track cart trajectory while balancing pendulum links.
   - **Methodology:** Extension of EIC forms for highly underactuated robots ($n < m$).

Approach 1: Periodic Orbit Stabilization (BVP-LQR)

System Transformation

- Let $\eta = [\vartheta_1, \vartheta_2, \vartheta_3]^T$. Degree of actuation: $m = 1$; Degree of underactuation: $n - m = 3$.
- By feedback linearizing the first row, cart dynamics is transformed into a double integrator:

$$\begin{cases} \ddot{x}_c = u \\ M(\eta)\ddot{\eta} + C(\eta, \dot{\eta})\dot{\eta} + g(\eta) + D\dot{\eta} + b(\eta)u = 0 \end{cases}$$

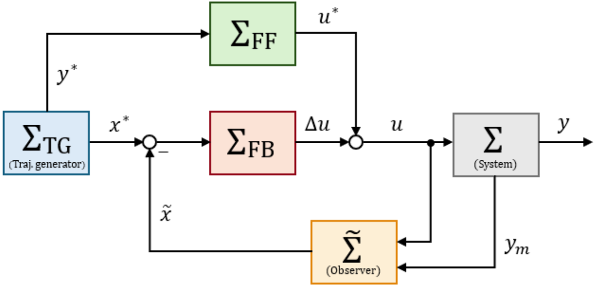
- The internal dynamics (3-dimensional subsystem) becomes:

$$\ddot{\eta} = \beta_f(\eta, \dot{\eta}) + \beta_g(\eta)u$$

where $\beta_f \in \mathbb{R}^3$ is the drift vector and $\beta_g \in \mathbb{R}^3$ is the input vector.

2-DOF Control Architecture

- **Feedforward** (Σ_{FF}): generates reference command based on internal dynamics.
- **Feedback** (Σ_{FB}): tracks reference based on real measurements.
- The resulting structure is a classical two-degrees-of-freedom architecture.



Feedforward Control via BVP

The feedforward controller solves a Two-Point Boundary Value Problem (BVP) over $t \in [0, T]$. Cart position is parameterized using sinusoidal basis functions:

$$x_c(t) = \Upsilon(t, p) = \sum_{i=1}^{2} a_i \sin\left(\frac{i\pi t}{T}\right) + \sum_{i=1}^{6} p_i \sin\left(\frac{(2+i)\pi t}{T}\right)$$

Differential Equation:

$$\ddot{\eta} = \beta_f(\eta, \dot{\eta}) + \beta_g(\eta)\ddot{\Upsilon}(t, p)$$

BVP Boundary Conditions

Solve for parameters $p^\star \in \mathbb{R}^6$ to satisfy boundary conditions:

Initial ($t = 0$):

$$x_c(0) = 0, \quad \dot{x}_c(0) = \dot{x}_{c,0}, \quad \eta_i(0) = 0, \quad \dot{\eta}_i(0) = \dot{\eta}_{i,0}$$

Terminal ($t = T$):

$$x_c(T) = 0, \quad \dot{x}_c(T) = \dot{x}_{c,0}, \quad \eta_i(T) \in \{0, 2\pi\}, \quad \dot{\eta}_i(T) = \dot{\eta}_{i,0}$$

Note: $\eta_i(T) = 2\pi$ signifies a full rotation, while $\eta_i(T) = 0$ signifies balancing.
The optimal cart acceleration is $u^\star(t) = \ddot{x}_c(t, p^\star)$.
Initial conditions on x_c are implied by the use of a sine series.

Feedback Control via TV-LQR

Let $x = [x_c, \eta^T, \dot{x}_c, \dot{\eta}^T]^T \in \mathbb{R}^8$. Linearizing along nominal trajectory $x^\star(t), u^\star(t)$:

$$\delta\dot{x} = A(t)\delta x + B(t)\delta u$$

Control computed to minimize the quadratic cost:

$$J(\delta x, \delta u) = \int_0^T \left(\delta x^T Q \delta x + \delta u^T R \delta u\right) dt$$

Riccati Equation & Optimal Gain

The optimal time-varying gain is derived from the Differential Riccati Equation:

$$-\dot{X} = XA + A^T X - XBR^{-1}B^T X + Q$$

Integrated backward in time from $X(T) = X_T$.

Periodic Solution

To enforce a *Positive Definite T-Periodic Matrix Solution*, integrate over multiple cycles to drive the gain toward a periodic steady state.

Final Control Law:

$$u(t) = u^{\star}(t) + K(t)\Big(x^{\star}(t) - x(t)\Big) \quad \text{where} \quad K(t) = -R^{-1}B^T(t)X(t)$$

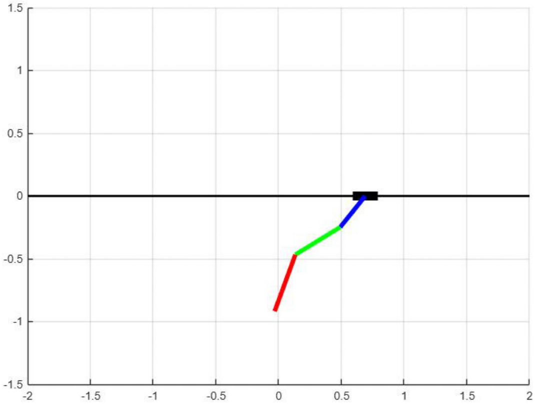
Simulations (1)

The complete routine consists of distinct maneuvers, each generated via BVP and stabilized with TV-LQR:

1. **Swing-up:** From stable downward equilibrium to unstable upward equilibrium.
2. **Entry:** Reaching the initialization state of the BVP.
3. **Periodic Motion:** Continuous tracking of the designed orbit.
4. **Defuse:** Return to upward equilibrium.
5. **Swing-down:** Return to stable downward rest.

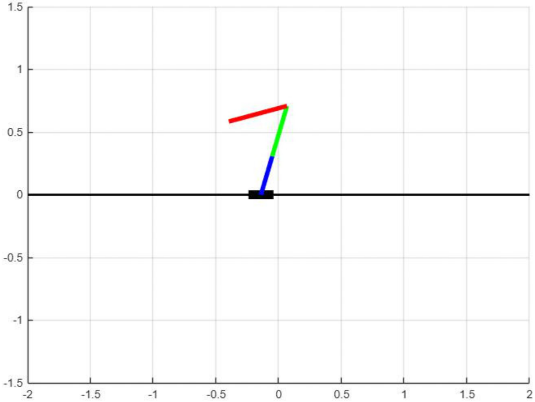
Simulations (2)

Swing-up + Entry + Periodic motion (5 cycles)



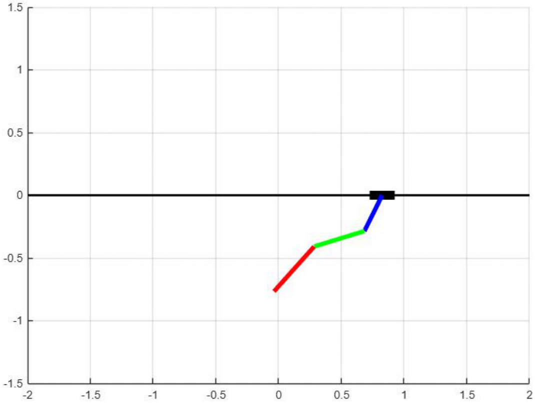
Simulations (3)

Defuse + Swing-down



Simulations (4)

Full simulation



Approach 2: Cascaded-EIC Control

Control goal

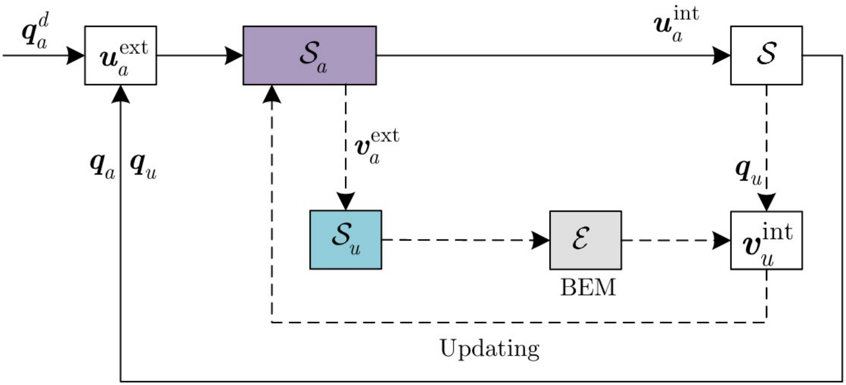
- Partition into actuated $q_a \in \mathbb{R}^n$ $(n = 1)$ and unactuated $q_u \in \mathbb{R}^m$ $(m = 3)$:

$$\begin{cases} D_{aa}\ddot{q}_a + D_{au}\ddot{q}_u + H_a = u \\ D_{ua}\ddot{q}_a + D_{uu}\ddot{q}_u + H_u = 0 \end{cases}$$

Control Goal

Track q_a^d (cart) while stabilizing zero dynamics q_u (pendulum).

EIC Framework



- **External loop:** Compute instantaneous equilibrium q_u^e.
- **Internal loop:** Update control to track q_u^e.

EIC Control: external loop

- Feedback linearization for actuated dynamics:

$$u_a^{ext} = D_{aa} v_a^{ext} + D_{au} \ddot{q}_u + H_a$$

$$\ddot{q}_a = v_a^{ext} = \ddot{q}_a^d + K_{d1}(\dot{q}_a^d - \dot{q}_a) + K_{p1}(q_a^d - q_a)$$

- Compute BEM (reference for unactuated coordinates):

$$\mathcal{E} = \{q_u^e : \Gamma = 0, \; \dot{q}_u = \ddot{q}_u = 0\}$$

$$\Gamma = D_{ua} v_a^{ext} + D_{uu} \ddot{q}_u + H_u$$

EIC Control: internal loop

- After BEM is computed:

$$\ddot{q}_u = v_u^{int} = \ddot{q}_u^e + K_{d2}(\dot{q}_u^e - \dot{q}_u) + K_{p2}(q_u^e - q_u)$$

- Virtual feedback linearization for unactuated dynamics:

$$v_a^{int} = -D_{ua}^{\dagger}(D_{uu} v_u^{int} + H_u)$$

- Control update:

$$u_a^{int} = D_{aa} v_a^{int} + D_{au}\ddot{q}_u + H_a$$

Cascaded-EIC Framework

Limit of EIC

Pseudo-inverse ($D_{ua}^{\dagger}$) loses information if $n < m$ (highly underactuated systems): recursively neste the dynamics until a fully actuated subsystem is reached.

- Isolate $\ddot{q}_a$ from the top level:

$$\ddot{q}_a = (D_{aa})^{-1}(u - D_{au}\ddot{q}_u - H_a)$$

- Plug into the unactuated dynamics to define the first subsystem $\mathcal{S}_1$:

$$(D_{uu} - D_{ua}D_{aa}^{-1}D_{au})\ddot{q}_u + (H_u - D_{ua}D_{aa}^{-1}H_a) = -D_{ua}D_{aa}^{-1}u$$

Subsystem $\mathcal{S}_1$

- Defining variables for $\mathcal{S}_1$:

$$
\begin{aligned}
q^{(1)} &= q_u \\
D^{(1)} &= D_{uu} - D_{ua} D_{aa}^{-1} D_{au} \\
H^{(1)} &= H_u - D_{ua} D_{aa}^{-1} H_a \\
B^{(1)} &= -D_{ua} D_{aa}^{-1}
\end{aligned}
$$

- A new (smaller) system is obtained:

$$
\mathcal{S}_1 : D^{(1)} \ddot{q}^{(1)} + H^{(1)} = B^{(1)} u
$$

- For $\mathcal{S}_1$, $\dim(q^{(1)}) = 3$ and inputs $n_1 = 1$: still an (highly) underactuated system.

Subsystem $\mathcal{S}_2$

- Partitioning $\mathcal{S}_1$ into new actuated/unactuated parts yields $\dim(q_a^{(1)}) = 1$, $\dim(q_u^{(1)}) = 2$.
- Iterating the substitution yields $\mathcal{S}_2$:

$$
\begin{aligned}
q^{(2)} &= q_u^{(1)} \\
D^{(2)} &= D_{uu}^{(1)} - D_{ua}^{(1)} (D_{aa}^{(1)})^{-1} D_{au}^{(1)} \\
H^{(2)} &= H_u^{(1)} - D_{ua}^{(1)} (D_{aa}^{(1)})^{-1} H_a^{(1)} \\
B^{(2)} &= B_u^{(1)} - D_{ua}^{(1)} (D_{aa}^{(1)})^{-1} B_a^{(1)}
\end{aligned}
$$

$$
\mathcal{S}_2 : D^{(2)} \ddot{q}^{(2)} + H^{(2)} = B^{(2)} u
$$

- For $\mathcal{S}_2$, $\dim(q^{(2)}) = 2$ and inputs $n_2 = 1$: still an underactuated system.

Subsystem $\mathcal{S}_3$: Fully Actuated Level

- Performing the procedure one last time:

$$
\begin{aligned}
q^{(3)} &= q_u^{(2)} \quad (\text{This is } \vartheta_3) \\
D^{(3)} &= D_{uu}^{(2)} - D_{ua}^{(2)} (D_{aa}^{(2)})^{-1} D_{au}^{(2)} \\
H^{(3)} &= H_u^{(2)} - D_{ua}^{(2)} (D_{aa}^{(2)})^{-1} H_a^{(2)} \\
B^{(3)} &= B_u^{(2)} - D_{ua}^{(2)} (D_{aa}^{(2)})^{-1} B_a^{(2)}
\end{aligned}
$$

$$
\mathcal{S}_3 : D^{(3)} \ddot{q}^{(3)} + H^{(3)} = B^{(3)} u
$$

Success

$\mathcal{S}_3$ is a scalar, fully actuated system.

Balance Equilibrium Manifold (BEM)

- For each subsystem, define BEM to find the reference equilibrium $q_u^{(i),e}$ for the unactuated variables.

$$\mathcal{E}_i = \left\{ q_a^{(i+1),e} : \Gamma_{i+1} = 0, \dot{q}_a^{(i+1)} = \ddot{q}_a^{(i+1)} = 0 \right\}$$

where

$$\Gamma_{i+1} = D_{aa}^{(i+1)} \ddot{q}_a^{(i+1)} + D_{au}^{(i+1)} \ddot{q}_u^{(i+1)} + H_a^{(i+1)} - B_a^{(i+1)} u$$

- Drive unactuated coordinates towards instantaneous equilibrium using a cascaded approach.

Control Design: External Loops

- **Subsystem $\mathcal{S}_0$ (cart):**

$$v_0^{ext} = \ddot{q}_a^{(0),d} + k_{d0}(\dot{q}_a^{(0),d} - \dot{q}_a^{(0)}) + k_{p0}(q_a^{(0),d} - q_a^{(0)})$$
$$u_0^{ext} = (B_a^{(0)})^{-1}\left(D_{aa}^{(0)} v_0^{ext} + D_{au}^{(0)} \ddot{q}_u^{(0)} + H_a^{(0)}\right)$$

- **Subsystems $\mathcal{S}_i,\ \mathbf{i = 1, 2}$:**

$$v_i^{ext} = \ddot{q}_a^{(i),d} + k_{di}\left(\dot{q}_a^{(i),d} - \dot{q}_a^{(i)}\right) + k_{pi}\left(q_a^{(i),d} - q_a^{(i)}\right)$$
$$u_i^{ext} = \left(B_a^{(i)}\right)^{-1}\left(D_{aa}^{(i)} v_i^{ext} + D_{au}^{(i)} \ddot{q}_u^{(i)} + H_a^{(i)}\right)$$

- **Subsystem $\mathcal{S}_3$ (fully actuated):**

$$v_3^{int} = \ddot{q}_a^{(3),d} + k_{d3}\left(\dot{q}_a^{(3),d} - \dot{q}_a^{(3)}\right) + k_{p3}\left(q_a^{(3),d} - q_a^{(3)}\right)$$
$$u_3^{int} = \left(B^{(3)}\right)^{-1}\left(D^{(3)} v_3^{int} + H^{(3)}\right)$$

Control Design: Internal Loops

- Control update:
- **Subsystems $\mathcal{S}_i$, i $= 2, 1$:**

$$v_i^{int} = \left(D_{aa}^{(i)}\right)^{-1}\left(B_a^{(i)} u_{i+1}^{int} - D_{au}^{(i)} \begin{bmatrix} v_{i+1}^{int} \\ \ddot{q}_u^{(i+1)} \end{bmatrix} - H_a^{(i)}\right)$$

$$u_i^{int} = \left(B_a^{(i)}\right)^{-1}\left(D_{aa}^{(i)} v_i^{int} + D_{au}^{(i)} \ddot{q}_u^{(i)} + H_a^{(i)}\right)$$

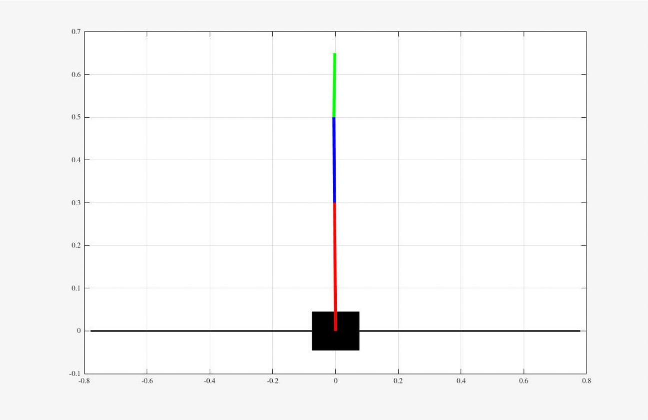
- **Subsystems $\mathcal{S}_0$ (final control law):**

$$v_0^{int} = \left(D_{aa}\right)^{-1}\left(u_1^{int} - D_{au} \begin{bmatrix} v_1^{int} \\ \ddot{q}_u^{(1)} \end{bmatrix} - H_a\right)$$

$$u_0^{int} = D_{aa} v_0^{int} + D_{au} \ddot{q}_u + H_a$$

Simulations

Tracking trajectory $q_a^{(0),d} = 0.5\sin(0.5t)$.



Conclusions

Project Outcomes:

- Successfully investigated control strategies for Triple Pendulum on a Cart.

Comparison of Approaches:

BVP-LQR Control

- Tool for generating and locally stabilizing specific, pre-designed periodic motions.
- Limited by local validity and reliance on very accurate model.

CEIC Control

- Nonlinear tool for simultaneous trajectory tracking and balancing in highly underactuated systems ($n < m$).
- Limited by complex computations for BEM and difficult calibration of controller gains.